Build Procedural Fluency from Conceptual Understanding

Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

Effective mathematics teaching focuses on the development of *both* conceptual understanding *and* procedural fluency. Major reports have identified the importance of an integrated and balanced development of concepts and procedures in learning mathematics (National Mathematics Advisory Panel 2008; National Research Council 2001). Furthermore, NCTM (1989, 2000) and CCSSM (NGA Center and CCSSO 2010) emphasize that procedural fluency follows and builds on a foundation of conceptual understanding, strategic reasoning, and problem solving.

Discussion

When procedures are connected with the underlying concepts, students have better retention of the procedures and are more able to apply them in new situations (Fuson, Kalchman, and Bransford 2005). Martin (2009, p. 165) describes some of the reasons that fluency depends on and extends from conceptual understanding:

> To use mathematics effectively, students must be able to do much more than carry out mathematical procedures. They must know which procedure is appropriate and most productive in a given situation, what a procedure accomplishes, and what kind of results to expect. Mechanical execution of procedures without understanding their mathematical basis often leads to bizarre results.

Fluency is not a simple idea. Being fluent means that students are able to choose flexibly among methods and strategies to solve contextual and mathematical problems, they understand and are able to explain their approaches, and they are able to produce accurate answers efficiently. Fluency builds from initial exploration and discussion of number concepts to using informal reasoning strategies based on meanings and properties of the operations to the eventual use of general methods as tools in solving problems. This sequence is beneficial whether students are building toward fluency with single- and multi-digit computation with whole numbers or fluency with, for example, fraction operations, proportional relationships, measurement formulas, or algebraic procedures.

Computational fluency is strongly related to number sense and involves so much more than the conventional view of it encompasses. Developing students' computational fluency extends far beyond having students memorize facts or a series of steps unconnected to understanding (Baroody 2006; Griffin 2005). A rush to fluency, however, undermines students' confidence and interest in mathematics and is considered a cause of mathematics anxiety (Ashcraft 2002; Ramirez et al. 2013). Further, early work with reasoning strategies is related to algebraic reasoning. As students learn how quantities can be taken apart and put back together in different ways (i.e., decomposition and composition of numbers), they establish a basis for understanding properties of the operations. Students need this early foundation for meaningful learning of more formal algebraic concepts and procedures throughout elementary school and into middle and high school (Carpenter, Franke, and Levi 2003; Griffin 2003; Common Core State Standards Writing Team 2011).

In meaningful learning of basic number combinations (i.e., addition and subtraction within 20 and multiplication and division within 100), students progress through well-documented phases toward fluency (Baroody 2006; Baroody, Bajwa, and Eiland 2009; Carpenter et al. 1999). Students begin by using objects, visual representations, and verbal counting, and then they progress to reasoning strategies using number relationships and properties. For example, to solve 8 + 4, a first grader might count on from 8 early in the school year, whereas later in the year the same student might reason that since 8 + 2 is 10, then 8 + 4 must be 2 more than 10, or 12. A third grader might initially use repeated addition to solve 4 × 6 and then progress to reason that 2 sixes are 12, so 4 sixes must be double that amount, which is 24. This approach supports students, over time, in knowing, understanding, and being able to use their knowledge of number combinations meaningfully in new situations.

Learning procedures for multi-digit computation needs to build from an understanding of their mathematical basis (Fuson and Beckmann 2012/2013; Russell 2000). For example, consider the work in figure 17 by David and Anna, two fourth graders, on a multiplication problem, 57 × 4 = □, and their explanations of what they have done.

David's solution	Anna's solution
+2 57 x 4 288	4 x 57 4 x 50 = 200 4 x 7 = 28 200 + 28 = 228
I multiplied 7 and 4 and got 28. I put down the 8 and carried the 2. Then I added the 2 and the 5 and got 7 and multiplied it by 4 and got 28. I put down the 28 and got 288.	I did it in parts. First I multiplied 4 x 50 and got 200. Then I multiplied 4 and 7 and got 28. Then I just added those two parts together to get the answer.

Fig. 17. David's and Anna's solutions to a multiplication problem. Adapted from Russell (2000).

David's faulty application of the multiplication algorithm leads to an incorrect answer that he should have recognized as too large (i.e., a reasonable answer must be less than 4×60). Anna's solution, by contrast, shows her understanding that 57 can be partitioned into tens and ones, that each quantity can be multiplied by 4 (an application of the distributive property), and that those new quantities can then be combined.

Similarly, a high school student who does not understand the distance formula,

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

may have trouble accurately recalling it and applying it appropriately to problem situations. By contrast, a student who understands that the formula is an application of the Pythagorean theorem (i.e., the distance between two points can be thought of as the hypotenuse of a right triangle) can use an understanding of this underlying relationship to solve a problem involving the distance between two points correctly (Martin 2009).

Clearly, students need procedures that they can use with understanding on a broad class of problems. This raises questions regarding *how* students can move most effectively toward fluency with general methods or algorithms, as well as what defines an algorithm. Fuson and Beckmann (2012/2013) argue that a standard algorithm is defined by its mathematical approach and not by the way in which the steps in the approach are recorded. They suggest that variations in written notation are not only acceptable but indeed valuable in supporting students' understanding of the base-ten system and properties of the operations. They also emphasize the importance of understanding, explaining, and visualizing: "Standard algorithms are to be understood and explained and related to visual models before there is any focus on fluency" (p. 28).

For example, as figure 18 illustrates, the conventional algorithm for multi-digit multiplication is difficult to understand, whereas the three alternative methods shown are more transparent with respect to the central mathematical features of place-value meanings and properties of the operations (Fuson 2003). The diagrams show the multiplication of tens and ones and the relative size (in area) of the partial products. The accessible algorithm shows a clear record of the four pairs of numbers that are multiplied. This progression also supports students in establishing a basis from which to apply and extend these understandings to operations with rational numbers and algebraic expressions.

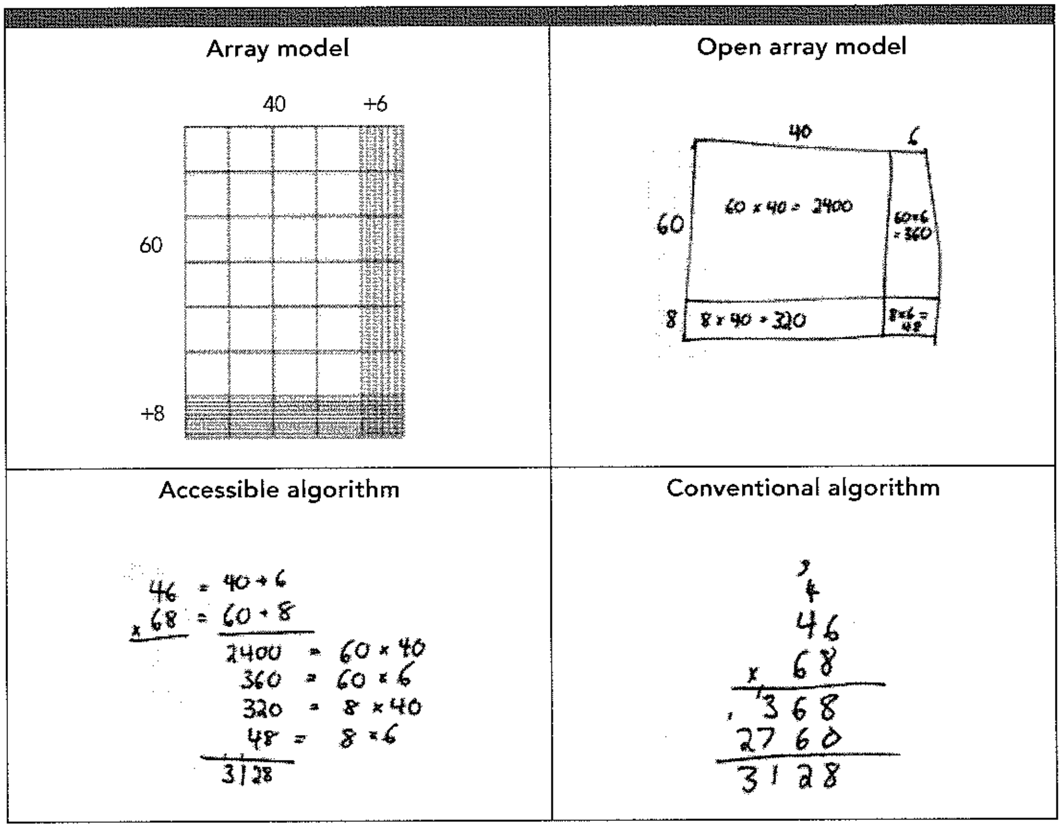

Fig. 18. Methods for multi-digit multiplication, using 68 × 46.
Adapted from Fuson (2003, p. 303).

In moving to fluency, students also need opportunities to rehearse or practice strategies and procedures to solidify their knowledge. However, giving students too many practice problems too soon is an ineffective approach to fluency. Students need opportunities to practice on a moderate number of carefully selected problems after they have established a strong conceptual foundation and the ability to explain the mathematical basis for a strategy or procedure. At that point, providing students with practice on a small number of problems, "spacing" or distributing these over time, and including feedback on student performance support learning outcomes (Pashler et al. 2007; Rohrer 2009; Rohrer and Taylor 2007).

Similarly, practice with basic number combinations should occur after students can explain and justify their use of efficient reasoning strategies. A word of caution is important in regard to timed tests. The premature and overuse of such tests may hinder students' mathematical proficiency and lower their confidence in themselves as learners of mathematics (Boaler 2012; Seeley 2009). Practice with basic number combinations should focus on solidifying students' use of an efficient strategy for specific number combinations (Rathmell 2005; Thornton 1978). Isaacs and Carroll (1999) suggest that practice be brief, engaging, purposeful, and distributed. For example, practice can target specific strategies, such as making a ten for addition or doubling a known fact for multiplication, and can be embedded in problem-solving tasks and games (Crespo, Kyriakides, and McGee 2005).

Illustration

Mr. Donnelly's use of the lesson featuring the Candy Jar task, illustrated in figure 13, is an important step in building his students' fluency in solving problems that involve proportional relationships. Mr. Donnelly helps his students understand that the ratios need to remain constant and that they can use different approaches to preserve this constant multiplicative relationship between the numerator and the denominator. Over time, Mr. Donnelly will need to discuss the efficiency of some strategies over others (e.g., using the scale factor is usually more efficient than scaling up by using a table), and he will need to provide examples of problems that specific strategies would be particularly useful in solving. Ultimately, Mr. Donnelly will want to give his students problems in which neither the unit rate nor the scale factor are integers (e.g., $5/13 = 127/x$) and ask students to devise methods for finding the missing value. Students might generate either of the approaches shown in figure 19, the scale factor method and the unit rate method.

Consider the reasoning that underlies each of these methods and how each is clearly grounded in an understanding of ratio concepts and multiplicative relationships. Mr. Donnelly could then ask his students to consider the generalizability of these approaches as another step toward fluency in solving problems involving proportional relationships.

Teacher and student actions

Effective teaching not only acknowledges the importance of both conceptual understanding and procedural fluency but also ensures that the learning of procedures is developed over time, on a strong foundation of understanding and the use of student-generated strategies in solving problems. This approach supports students in developing the ability to understand and explain their use of procedures, choose flexibly among methods and strategies to solve contextual and mathematical problems, and produce accurate answers efficiently. The actions identified in the table at the right summarize what teachers and students are doing in mathematics classrooms to build procedural fluency from conceptual understanding and problem-solving experiences.

$\frac{5}{13} = \frac{127}{x}$	
Scale factor method	**Unit rate method**
JR: $5n = 127$ $n = 25.4$ ← scale factor JB: $13 \cdot 25.4 = 330.2$	$5n = 13$ Unit rate $= 2.6$ So 1 JR $= 2.6$ JB $127 \cdot 2.6 = 330.2$ jaw breakers in the new jar
Student explanation: "The original jar contained 5 Jolly Ranchers, but the new jar contains 127 Jolly Ranchers, so 5 times some number is 127. So, 127 ÷ 5 = 25.4. So, this is the factor that I need to use because the new jar has to have 25.4 times more Jolly Ranchers. Since the original jar had 13 jawbreakers and I need to keep the same ratio, I needed to multiply 13 by the same scale factor, so 13 × 25.4 = 330.2 jawbreakers in the new jar."	**Student explanation:** "The ratio is 5 Jolly Ranchers for every 13 jawbreakers, so 5 times some number is 13. If I distribute the 13 jawbreakers equally among the 5 Jolly Ranchers, 13 ÷ 5 = 2.6, which gives the ratio of 1 Jolly Rancher for every 2.6 jawbreakers, so 2.6 is the unit rate. Since I have 127 Jolly Ranchers, or units, in the new jar, I have to multiply this by the unit rate, so 127 × 2.6 = 330.2 jawbreakers. "Well, 330.2 is the exact answer. But since jawbreakers have to be whole numbers, the answer to problem is 330 jawbreakers."

Fig. 19. Student approaches to the Candy Jar task, leading to general methods

Build procedural fluency from conceptual understanding **Teacher and student actions**	
What are *teachers* doing?	**What are *students* doing?**
Providing students with opportunities to use their own reasoning strategies and methods for solving problems. Asking students to discuss and explain why the procedures that they are using work to solve particular problems. Connecting student-generated strategies and methods to more efficient procedures as appropriate.	Making sure that they understand and can explain the mathematical basis for the procedures that they are using. Demonstrating flexible use of strategies and methods while reflecting on which procedures seem to work best for specific types of problems. Determining whether specific approaches generalize to a broad class of problems.

Build procedural fluency from conceptual understanding Teacher and student actions, *continued*	
What are *teachers* doing?	What are *students* doing?
Using visual models to support students' understanding of general methods. Providing students with opportunities for distributed practice of procedures.	Striving to use procedures appropriately and efficiently.

Support Productive Struggle in Learning Mathematics

Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

Effective mathematics teaching supports students in struggling productively as they learn mathematics. Such instruction embraces a view of students' struggles as opportunities for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions. In contrast to productive struggle, unproductive struggle occurs when students "make no progress towards sense-making, explaining, or proceeding with a problem or task at hand" (Warshauer 2011, p. 21). A focus on student struggle is a necessary component of teaching that supports students' learning of mathematics with understanding (Hiebert and Grouws 2007). Teaching that embraces and uses productive struggle leads to long-term benefits, with students more able to apply their learning to new problem situations (Kapur 2010).

Discussion

In comparisons of mathematics teaching in the United States and in high-achieving countries, U.S. mathematics instruction has been characterized as rarely asking students to think and reason with or about mathematical ideas (Banilower et al. 2006; Hiebert and Stigler 2004). Teachers sometimes perceive student frustration or lack of immediate success as indicators that they have somehow failed their students. As a result, they jump in to "rescue" students by breaking down the task and guiding students step by step through the difficulties. Although well intentioned, such "rescuing" undermines the efforts of students, lowers the cognitive demand of the task, and deprives students of opportunities to engage fully in making sense of the mathematics (Reinhart 2000; Stein et al. 2009). As teachers plan lessons, key components for them to consider are the student struggles and misconceptions that might